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Operations in the Space $\sigma'\{M_p(x,q)\}$

by

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Abstract

We investigate: the operation of multiplying exponential ultradistributions from σ' by ultradifferentiable functions; the operation of differentiation in σ'; operations of direct product and convolution in the spaces $\mathscr{D}'^{(N_q)}$, $\mathscr{E}'^{(N_q)}$ and σ'.

Introduction

In this paper we shall continue to investigate spaces of exponential ultradistributions $\sigma'\{M_p(x,q)\}$ (abbr. σ') introduced in [6]. We shall give the sufficient conditions which assure that the multiplication of elements from $\sigma'\{M_p(x,q)\}$ by ultradifferentiable functions is a continuous mapping into itself. (A smooth function is called ultradifferentiable if it belongs to some space $\mathscr{E}^{(N_q)}$). We shall give conditions on the coefficients of an ultradifferentiable operator under which it acts continuously on $\sigma'\{M_p(x,q)\}$.

We shall give the definition of the direct product in the spaces $\mathscr{D}'^{(N_q)}$, $\mathscr{E}'^{(N_q)}$ and the space σ'. We shall give as well the definition on the convolution in these spaces. Our definition is similar to the definition of the convolution in the space $\mathscr{D}'$ of Schwartz distributions given by Vladimirov [8]. We call the convolution in σ', σ'-convolution, and for this convolution we use a symbol $\overset{\sigma}{*}$. If $f,g\in\sigma'\subset\mathscr{D}'^{(N_q)}$, then it is possible for the convolution $f*g$ in $\mathscr{D}'^{(N_q)}$ to exist, but the σ-convolution $f\overset{\sigma}{*}g$ does not exist.

Many classical properties of the convolution easily follow from our definitions of the convolution.

Basic notions and notations

We shall use the same notation as in [6]. $\{N_q;\ q\in N_0\}(N_0=N\cup\{0\})$ will denote a sequence of positive numbers for which we suppose (as in [2]):

(M.1) $$N_q^2\le N_{q-1}N_{q+1}\,,\qquad q\in N;$$

(M.2)′ There are constants A and H such that

$$N_{q+1}\le AH^qN_q\,,\qquad q\in N_0\,;$$

(M.3)′ $$\sum_{q=1}^{\infty} N_{q-1}/N_q < \infty .$$

In some assertions instead of (M.2)′ we shall assume the stronger one:

(M.2) There are constants A and H such that

$$N_q \leq AH^q \min_{0 \leq i \leq q} N_i N_{q-i}, \qquad q \in \boldsymbol{N}_0 .$$

$\{C_{p,q};\ (p, q) \in \boldsymbol{N} \times \boldsymbol{N}_0\}$ will denote a matrix of positive numbers for which we assume:

(C.1) For every $p \in \boldsymbol{N}$, the sequence $\{C_{p,q};\ q \in \boldsymbol{N}_0\}$ monotonically tends to zero when $q \to \infty$;

(C.2) For every $p \in \boldsymbol{N}$ there is $p' \in \boldsymbol{N}$ such that for every $\varepsilon > 0$ there exists $q_0(\varepsilon) \in \boldsymbol{N}_0$ with the property

$$C_{p,q} \leq \varepsilon C_{p',q} \qquad \text{for} \quad q \geq q_0(\varepsilon) .$$

(C.3) For every $p \in \boldsymbol{N}$ there is $p' \in \boldsymbol{N}$ such that

$$\sup_{q \in \boldsymbol{N}_0} \{C_{p,q}/C_{p',q+1}\} < \infty .$$

In Theorem 5 we shall use the condition stronger than (C.2):

(C.4) For every $p \in \boldsymbol{N}$ there are $p' \in \boldsymbol{N}$ and $A_{p,p'} > 0$ suc that

$$2^q C_{p,q} \leq A_{p,p'} C_{p',q}, \qquad q \in \boldsymbol{N}_0 .$$

In the definition of the direct product we shall use the condition much more stronger than (C.3):

(C.5) for every $p \in \boldsymbol{N}$ there is $p_0' \in \boldsymbol{N}$, $p_0' > p$, such that for every $p' \geq p_0'$ there is $K_{p,p'} > 0$ such that

$$\max_{0 \leq \alpha \leq q} \{C_{p,\alpha} C_{p,q-\alpha}\} \leq K_{p,p'} C_{p',q} .$$

As in [6] we put

$$m_p(x) = m_{1,p}(x_1) + \cdots + m_{s,p}(x_s), \qquad p \in \boldsymbol{N}, \qquad (x = (x_1, \cdots, x_s) \in \boldsymbol{R}^s)$$

where $m_{i,p}(x_i)$, $i = 1, \cdots, s$, are convex even functions constructed in [6]. These functions increase to infinity faster than any linear function when $x_i \to \infty$. We also assume:

(A) for every $p \in \boldsymbol{N}$ there are $p_i' \in \boldsymbol{N}$, $i = 1, \cdots, s$, such that

$$m_{i,p}(px_i) \leq m_{i,p_i'}(x_i) \qquad \text{for} \quad x_i \geq p_i', \qquad i = 1, \cdots, s .$$

Our construction of the sequence $m_p(x)$, $p \in \boldsymbol{N}$ is connected with the Fourier transformation of spaces σ and σ'.

Spaces $\mathscr{E}^{(N_q)}$ and $\mathscr{D}^{(N_q)}$ ($\mathscr{E}^{(N_q)}(\boldsymbol{R}^s) = \mathscr{E}^{(N_q)}$, $\mathscr{D}^{(N_q)}(\boldsymbol{R}^s) = \mathscr{D}^{(N_q)}$) are defined and deeply investigated in [2].

A smooth function ϕ belongs to $\mathscr{E}^{(N_q)}$ iff for every regular compact set $K \subset \boldsymbol{R}^s$ and every $p \in \boldsymbol{N}$

$$\text{(1)} \qquad \|\phi\|_{(N_q),K,p} := \sup_{\substack{x \in K \\ q \in N_0{}^s}} \frac{p^{|q|} |D^q \phi(x)|}{N_{|q|}} < \infty .$$

A regular compact set in $\boldsymbol{R}^s$ is a compact one with a finite number of connected components each having the following property (of Whitney) ([2]):

There is a number C such that any two points x and y of connected component L are joined by an arc in L of length less than or equal to $C|x-y|$.

The topology in $\mathscr{E}^{(N_q)}$ is a projective one defined by the sequence of norms $\{\| \ \|_{(N_q),K_p,p}\}$, where $K_p = B(0, p)$ are closed balls with centers in zero and with radiuses p, $p \in \boldsymbol{N}$.

Space $\mathscr{D}_K^{(N_q)}$ is defined as a space of all $\phi \in C^\infty$ which satisfy (1) for all $p \in \boldsymbol{N}$ with a support in K, K is a regular compact subset of $\boldsymbol{R}^s$. With the norm $\| \ \|_{(N_q),K,p}$, $\mathscr{D}_K^{(N_q)}$ is a closed subspace of $\mathscr{E}^{(N_q)}$. Space $\mathscr{D}^{(N_q)}$ is an inductive limit space of the inductive sequence $\{\mathscr{D}_{K_p}^{(N_q)}; K_p = B(0, p)\}$. Condition (M.3)$'$ (besides (M.1)) is necessary and sufficient so as for any compact set $K \subset \boldsymbol{R}^s$ there exists $\phi \in \mathscr{D}^{(N_q)}$ such that $\phi \equiv 1$ on K.

We shall use the following notation:

$$\text{(2)} \quad M_p(x, q) = C_{p,q} \exp(m_p(x)), \qquad p \in \boldsymbol{N}, \qquad q \in \boldsymbol{N}_0, \qquad \text{abbreviated} \quad M_p .$$

The space $\sigma\{M_p(x, q)\}$ (abbreviated $\sigma\{M_p\}$ or σ) is defined ([6]) as a subspace of C^∞ such that

$$\phi \in \sigma \quad \text{iff} \quad \gamma_p(\phi) := \sup_{\substack{x \in \boldsymbol{R}^s \\ q \in N^s}} \{C_{p,|q|} |D^q \phi(x)| \exp(m_p(x))\} < \infty , \qquad p \in \boldsymbol{N}.$$

The projective topology in σ is defined by the sequence of norms $\{\gamma_p; p \in \boldsymbol{N}\}$. We give in [6] the structural properties of the space σ and its dual σ'.

If $C_{p,q}$, $(p, q) \in \boldsymbol{N} \times \boldsymbol{N}_0$, are of the form

$$\text{(3)} \qquad C_{p,q} = p^q / N_q ,$$

then $\mathscr{D}^{(N_q)} \hookrightarrow \sigma\{M_p(x, q)\}$ ([6]). The notation $X \hookrightarrow Y$ means that X is a dense subspace of Y and that the inclusion mapping of X into Y is continuous. We also proved in [6] that $\mathscr{D}^{(N_q)} \hookrightarrow \sigma\{M_p(x, q)\}$ if:

(4) for every $p \in \boldsymbol{N}$ there are $p' \in \boldsymbol{N}$, $p' > p$, and $D_{p,p'} > 0$ such that

$$2^q C_{p,q} \leq D_{p,p'} C_{p',q-r} (p')^r / N_r , \qquad r \in \boldsymbol{N}_0 , \qquad q \in \boldsymbol{N}_0 \quad \text{and} \quad r \leq q .$$

Since $\mathscr{D}^{(N_q)}$ is a dense subspace of $\mathscr{E}^{(N_q)}$ if (M.3)$'$ and (M.1) hold, we have

PROPOSITION 1. *If* (4) *holds then* $\sigma\{M_p\} \hookrightarrow \mathscr{E}^{(N_q)}$.

In our consideration the following condition is needed as well:

(5) for every $p \in N$ there exist $p' \in N$ and $M_{p,p'} > 0$ such that for every $q \in N_0$ and $r \in N_0$, $r \leq q$,

$$p^{q-r} N_r / N_q \leq M_{p,p'} C_{p',q-r}$$

Let us remark that conditions (4) and (5) imply together

$$\{\sigma(p^q/N_q) \exp(m_p(x))\} \equiv \sigma\{C_{p,q} \exp(m_p(x))\}.$$

Multiplication by ultradifferentiable function and differentiation

Similarly as in [2], we shall prove

THEOREM 2. *Let P_1 (resp. P_2, resp. P, resp. P_3) denotes the bilinear mapping $(\phi, \psi) \to \phi \cdot \psi$ from $\mathscr{D}^{(N_q)} \times \sigma\{M_p\}$ into $\mathscr{D}^{(N_q)}$ (resp. $\mathscr{D}^{(N_q)} \times \sigma\{M_p\}$ into $\sigma\{M_p\}$, resp. $\sigma\{M_p\} \times \sigma\{M_p\}$ into $\sigma\{M_p\}$, resp. $\mathscr{E}^{(N_q)} \times \sigma\{M_p\}$ into $\mathscr{E}^{(N_q)}$). Then*

(i) *The mapping P_1 is continuous if* (5) *holds.*

(ii) *The mapping P_2 is continuous if* (4) *holds.*

(iii) *If the following condition also holds:*

(6) *for every $p \in N$ there exist $p' \in N$ and $L_{p,p'} > 0$ such that*

$$2^q C_{p,q} \leq L_{p,p'} C_{p',q-r} C_{p',r}, \qquad r \in N_0, \qquad q \in N_0, \qquad r \leq q,$$

then the mapping P is continuous.

(iv) *The mapping P_3 is continuous if* (5) *holds.*

Proof. (i) Let $\phi \in \mathscr{D}^{(N_q)}$, supp $\phi \subset K$ and let $\psi \in \sigma\{M_p(x, q)\}$. For every $p \in N$ we have

$$\sup_{\substack{x \in K \\ q \in N_0{}^s}} \left\{ \frac{(p/2)^{|q|}}{N_{|q|}} |D^q(\phi(x)\psi(x))| \right\} = \sup_{\substack{x \in K \\ q \in N_0{}^s}} \left\{ 2^{-|q|} \sum_{i \leq q} \binom{q}{i} \frac{p^{|i|}}{N_{|i|}} |D^i \phi(x)| \right\}$$

$$\times C_{p_j, |q-i|} |D^{q-i}\psi(x)| \exp(m_p(x)) \frac{N_{|i|} p^{|q-i|}}{N_{|q|} C_{p',|q-i|}}.$$

Condition (5) implies that

$$\frac{N_{|i|} p^{|q-i|}}{N_{|q|} C_{p',|q-i|}} \leq M_{p,p'}, \qquad i \leq q, \qquad (i \leq q \text{ means } i_j \leq q_j, j = 1, \cdots, s).$$

So we obtain

$$\|\phi\psi\|_{(N_q), K, p/2} \leq M_{p,p'} \|\phi\|_{(N_q), K, p} \gamma_{p'}(\psi).$$

Thus we proved (i). The proofs of (ii), (iii) and (iv) are similar to the proof of (i).

In [2] it was proved that the spaces $\mathscr{E}^{(N_q)}$ and $\mathscr{D}^{(N_q)}$ are barrelled, bornologic, Montel and therefore reflexive spaces. Since the space $\sigma\{M_p\}$ is an *FS* space ([6]) it follows that it is barrelled, bornologic, Montel and therefore reflexive. Strong dual of a Montel space is also a Montel space so $\mathscr{D}'^{(N_q)}$, $\mathscr{E}'^{(N_q)}$, $\sigma'\{M_p\}$ are Montel spaces.

Theorem 2 implies the following theorem.

THEOREM 3. *Let* $\tilde{P}_1$ *(resp.* $\tilde{P}_2$*, resp.* $\tilde{P}$*, resp.* $\tilde{P}_3$*) denotes the bilinear mapping* $(\phi, f) \longmapsto \phi \cdot f$ *from* $\sigma\{M_p\} \times \mathscr{D}'^{(N_q)}$ *into* $\mathscr{D}'^{(N_q)}$ *(resp.* $\mathscr{D}^{(N_q)} \times \sigma'\{M_p\}$ *into* $\sigma'\{M_p\}$*, resp.* $\sigma\{M_p\} \times \sigma'\{M_p\}$ *into* $\sigma'\{M_p\}$*, resp.* $\sigma\{M_p\} \times \mathscr{E}'^{(N_q)}$ *into* $\mathscr{E}'^{(N_q)}$*) defined by*

$$(7) \qquad \langle \phi f, \psi \rangle := \langle f, \phi\psi \rangle, \ \psi \in \mathscr{D}^{(N_q)},$$

($\psi \in \sigma\{M^p\}$*, resp.* $\psi \in \sigma\{M_p\}$*, resp.* $\psi \in \mathscr{E}^{(N_q)}$*). Then*

(i) *The mapping* $\tilde{P}_1$ *is hypocontinuous if* (5) *holds.*
(ii) *The mapping* $\tilde{P}_2$ *is hypocontinuous if* (4) *holds.*
(iii) *The mapping* $\tilde{P}$ *is hypocontinuous if* (6) *holds.*
(iv) *The mapping* $\tilde{P}_3$ *is hypocontinuous if* (5) *holds.*

Proof. Since all the spaces mentioned in Theorem 3 are barrelled, if we prove that the mappings $\tilde{P}_1$, $\tilde{P}_2$, $\tilde{P}$ and $\tilde{P}_3$ are separately continuous, then from [7] Theorem 41.2. (p. 421) it follows that $\tilde{P}_1$, $\tilde{P}_2$, $\tilde{P}$ and $\tilde{P}_3$ are hypocontinuous.

We shall prove only the assetrion (i) because the other assertions can be proved in the same way.

(i) From Theorem 2(i) it follows that the mapping from $\mathscr{D}'^{(N_q)}$ into $\mathscr{D}'^{(N_q)}$ defined by (7) for a fixed $\phi \in \sigma\{M_p\}$ is continuous with respect to the strong topologies because this mapping is the adjoined one for the corresponding continuous mapping of $\mathscr{D}^{(N_q)}$ into $\mathscr{D}^{(N_q)}$. In order to prove that for a fixed $f \in \mathscr{D}'^{(N_q)}$ the mapping of $\sigma\{M_p\}$ into $\mathscr{D}'^{(N_q)}$ is continuous we shall use the fact that $\sigma\{M_p\}$ and $\mathscr{D}'^{(N_q)}$ are bornologic. (It is proved in [2] Theorem 5.4, that the space $\mathscr{D}'^{(N_q)}$ is bornologic).

If we prove that for every bounded subset B of σ, the set $B_1 = \{\phi f;\ \phi \in B\}$ is bounded in $\mathscr{D}'^{(N_q)}$ we shall obtain the assertion which is to be proved.

From Theorem 2(i) it follows that for any bounded subset B_2 of $\mathscr{D}^{(N_q)}$, the set $\{\phi\psi;\ \phi \in B,\ \psi \in B_2\}$ is bounded in $\mathscr{D}^{(N_q)}$. Therefore,

$$\sup \{ | \langle f\phi, \psi \rangle |;\ \phi \in B,\ \psi \in B_2 \} < \infty$$

and thus we obtain the assertion.

We shall now investigate the differentiation in $\sigma'\{M_p\}$.

From condition (C.3) it easily follows that the differentiation is a continuous operation in $\sigma\{M_p\}$. It means that if we define the differentiation in $\sigma'\{M_p\}$ as it is usual by

$$\langle Df, \phi \rangle := -\langle f, D\phi \rangle, \qquad \phi \in \sigma,$$

then it is a continuous mapping of σ' into σ' with respect to strong topologies. If the elements of $\{C_{p,q}\}$ are of the form (3), then (C.3) implies condition (M.2)$'$.

We denote by

$$P(D) = \sum_{\beta \in N_0{}^s} a_\beta(x) D^\beta, \qquad a_\beta(x) \in C^\infty(\boldsymbol{R}^s), \quad \beta \in N_0^s,$$

a differential operator which acts formally on $\phi \in C^\infty$ in the following way

$$P(D)\phi(x) := \sum_{\beta\in N_0^s} a_\beta(x) D^\beta \phi(x)\,. \tag{8}$$

The question we want to answer (at least partially) is what conditions on $\{C_{p,q}\}$ and $\{a_\beta;\ \beta\in N_0^s\}$ we have to assume in order to obtain the continuity of the mapping $P(D)\colon \sigma\{M_p\}\to\sigma\{M_p\}$.

THEOREM 4. *Let $\{C_{p,q}\}$ be of the form* (3), *and let $\{N_q\}$ satisfy conditions* (M.2) *and*

(9) *For every $p\in N$ there are $K_p>0$ and $p_1\in N$ such that*

for every $\beta\in N_0^s$ and $x\in R^s$

$$|D^q a_\beta(x)| \le K_p(p^{|\beta|}2^{-|q|}/N_{|\beta|})\exp(m_{p_1}(x)),\qquad q\in N_0^s\,.$$

Then the mapping $\phi\longmapsto P(D)\phi$ from $\sigma\{M_p\}$ into itself is continuous.

We note that (M.1), (M.2)′ and (M.3)′ hold for $\{N_q\}$.

Proof. From (9) and (A) it follows that there are $p_0\in N$ and $C>0$ such that

$$\gamma_p\Big(\sum_{\beta\in N_0^s} a_\beta(x)\phi^{(\beta)}(x)\Big) \le \sum_{\beta\in N_0^s}\sup_{\substack{x\in R^s\\ q\in N_0^s}}\left\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x))\sum_{r\le q}\binom{q}{r}|a_\beta^{(q-r)}(x)\phi^{(\beta+r)}(x)|\right\}$$

$$\le K_p\sum_{\beta\in N_0^s}\sup_{\substack{x\in R^s\\ q\in N_0^s}}\left\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x)+m_{p_1}(x))p^{|\beta|}2^{-|q|}\sum_{r\le q}\binom{q}{r}\frac{2^{|r|}}{N_{|\beta|}}|\phi^{(\beta+r)}(x)|\right\}$$

$$\le AK_p\sum_{\beta\in N_0^s}2^{-|\beta|}\sup_{\substack{x\in R^s\\ q\in N_0^s}}\left\{\frac{p^{|q|}}{N_{|q|}}\exp(m_p(x)+m_{p_1}(x))2^{-|q|}\right.$$

$$\left.\times\sum_{r\le q}\binom{q}{r}\frac{N_{|r|}}{p^{|r|}}\frac{(4pH)^{|\beta+r|}|\phi^{(\beta+r)}(x)|}{N_{|\beta+r|}}\right\}$$

$$\le C\gamma_{p_0}(\phi)\sum_{\beta\in N_0^s}2^{-|\beta|}\sup_{q\in N_0^s}\left\{\frac{p^{|q|}}{N_{|q|}}2^{-|q|}\sum_{r\le q}\binom{q}{r}\frac{N_{|r|}}{p^{|r|}}\right\}.$$

Since (M.3)′ holds we have that for a given $p\in N$ there is $S_p>0$ such that

$$\frac{N_q}{N_r}=\frac{N_qN_{q-1}\cdots N_{r+1}}{N_{q-1}N_{q-2}\cdots N_r}\ge S_p p^{q-r}\qquad \text{for every}\quad q,r\in N_0,\quad r\le q\,.$$

Thus by using $N_r/p^r\le(1/S_p)N_q/p^q$, $p,r\in N_0$, $r\le p$, we obtain that there is $C_1>0$ such that

$$\gamma_p\Big(\sum_{\beta\in N_0^s.} a_\beta(x)\phi^{(\beta)}(x)\Big)\le C_1\gamma_{p_0}(\phi)\,.$$

EXAMPLE. Suppose $m_p(x)\ge x/2$, $x\in R$, $p\in N$. We put

$$a_\beta(x)=\frac{x}{N_\beta}, \quad \beta \in N_0 \backslash \Lambda, \qquad a_\beta(x)=\frac{\exp(x/2)}{N_\beta}, \quad \beta \in \Lambda$$

where Λ is arbitrary subset of N_0. Condition (9) holds for this sequence.

THEOREM 5. *Let the matrix $\{C_{p,q}\}$ satisfy conditions* (C.4), (C.5) *and*

(10) *For every $p \in N$ there are $D_p > 0$ and $p_1 \in N$ such that for every $x \in \boldsymbol{R}^s$ and every $\beta \in N_0^s$*

$$|a_\beta^{(q)}(x)| \leq D_p C_{p,|\beta|} 2^{-|q|} \exp(m_{p_1}(x)), \qquad q \in N_0^s.$$

Then by $\phi \longmapsto P(D)\phi$ a continuous mapping from $\sigma\{M_p\}$ into $\sigma\{M_p\}$ is defined.

The proof of this theorem is similar to the preceding proof so we omit it.

From the last two theorems we obtain sufficient conditions on $\{C_{p,q}\}$ and $\{a_\beta(x); \beta \in N_0^s\}$ under which the mapping from $\sigma'\{M_p\}$ into $\sigma'\{M_p\}$ defined by

$$\left\langle \sum_{\beta \in N_0{}^s} (-1)^{|\beta|} (a_\beta(x) f(x))^{(\beta)}, \phi(x) \right\rangle = \left\langle f(x), \sum_{\beta \in N_0{}^s} a_\beta(x) \phi^{(\beta)}(x) \right\rangle, \quad \phi \in \sigma\{M_p\},$$

as adjoined mapping to the mapping from $\sigma\{M_p\}$ into $\sigma\{M_p\}$ defined by (7), is continuous with respect to the strong topologies.

Direct product

Up to the end of the paper we shall suppose that conditions (M.2) and (C.5) hold for $\{N_q\}$ and $\{C_{p,q}\}$.

To distinguish the subspaces of ultradistributions on $\boldsymbol{R}^s$ and $\boldsymbol{R}^{s+t}$ we shall use the following notations:

$$\mathscr{D}^{(N_q)}(\boldsymbol{R}^s), \quad \mathscr{D}^{(N_q)}(\boldsymbol{R}^{s+t}), \quad \sigma\{M_p\}(\boldsymbol{R}^s), \quad \sigma\{M_p\}(\boldsymbol{R}^{s+t}), \cdots$$

and

$$\mathscr{D}_{x,y}^{(N_q)} = \mathscr{D}^{(N_q)}(\boldsymbol{R}^{s+t}), \quad \mathscr{D}_x^{(N_q)} = \mathscr{D}^{(N_q)}(\boldsymbol{R}^s), \quad \sigma_{x,y}\{M_p\} = \sigma\{M_p\}(\boldsymbol{R}^{s+t}), \cdots$$

If $\phi(x, y) \in \mathscr{D}^{(N_q)}(\boldsymbol{R}^{s+t})(\phi(x, y) \in \mathscr{E}^{(N_q)}(\boldsymbol{R}^{s+t}))$, then ϕ can be observed as the family of ultradifferentiable functions from $\mathscr{D}^{(N_q)}(\boldsymbol{R}^t)$ $(\mathscr{E}^{(N_q)}(\boldsymbol{R}^t))$

$$\{\phi(x, y);\ x \in \boldsymbol{R}^s\}.$$

Thus by

$$x \longmapsto \psi(x) = \langle g(y), \phi(x, y) \rangle, \qquad g \in \mathscr{D}_y'^{(N_q)} \qquad (g \in \mathscr{E}_y'^{(N_q)}) \tag{11}$$

a function on $\boldsymbol{R}^s$ is defined.

THEOREM 6. *Let $\phi(x, y) \in \mathscr{D}_{x,y}^{(N_q)}$ $(\phi(x, y) \in \mathscr{E}_{x,y}^{(N_q)})$ and let $g \in \mathscr{D}_y'^{(N_q)}$ $(g \in \mathscr{E}_y'^{(N_q)})$. The function ψ defined by* (11) *belongs to $\mathscr{D}_x^{(N_q)}$ $(\mathscr{E}_x^{(N_q)})$ and the mapping from $\mathscr{D}_{x,y}^{(N_q)} \times \mathscr{D}_y'^{(N_q)}(\mathscr{E}_{x,y}^{(N_q)} \times \mathscr{E}_y'^{(N_q)})$ into $\mathscr{D}_x^{(N_q)}(\mathscr{E}_x^{(N_q)})$ defined by*

$$(g, \phi) \longmapsto \psi \tag{12}$$

is hypocontinuous.

Proof. Using the parts of the proof of Theorem 6.10 from [2], we obtain that ψ belongs to $\mathscr{D}_x^{(N_q)}(\mathscr{E}_x^{(N_q)})$. To prove that the mapping (12) is hypocontinuous it is enough to prove that this mapping is separately continuous. We shall prove it for the spaces $\mathscr{D}_{x,y}^{(N_q)}$ and $\mathscr{D}_y^{\prime(N_q)}$:

Let $g \in \mathscr{D}_y^{\prime(N_q)}$ and $p \in \boldsymbol{N}$ be fixed. If $\operatorname{supp} \phi = K$, then $K \subset K_1 \times K_2$ where K_1 and K_2 are suitable compact subsets of $\boldsymbol{R}^s$ and $\boldsymbol{R}^t$. Since $g \in \mathscr{D}_y^{\prime(N_q)}$, for any compact set $K \in \boldsymbol{R}^t$, there are $C > 0$ and $p \in \boldsymbol{N}$ such that $|\langle g, \theta \rangle| < C\|\theta\|_{(N_q), K, p}$, $\theta \in \mathscr{D}_y^{(N_q)}$.

So, there are $C > 0$ and $p_1 \in \boldsymbol{N}$ such that

$$\begin{aligned}
\|\psi\|_{(N_q), K_1, p} &= \sup_{\substack{x \in K_1 \\ \alpha \in \boldsymbol{N}_0{}^s}} \left\{ \frac{p^{|\alpha|}}{N_{|\alpha|}} |\langle g(y), D_x^\alpha \phi(x, y) \rangle| \right\} \\
&\leq C \sup_{\substack{x \in K_1 \\ \alpha \in \boldsymbol{N}_0{}^s}} \left\{ \frac{p^{|\alpha|}}{N_{|\alpha|}} \sup_{\substack{y \in K_2 \\ \beta \in \boldsymbol{N}_0{}^t}} \left\{ \frac{p_1^{|\beta|}}{N_{|\beta|}} |D_x^\alpha D_y^\beta \phi(x, y)| \right\} \right\} \\
&\leq AC \sup_{\substack{(x, y) \in K \\ \alpha \in \boldsymbol{N}_0{}^{s+t}}} \left\{ \frac{p_2^{|\alpha|}}{N_{|\alpha|}} |D^\alpha \phi(x, y)| \right\} \\
&= AC \|\phi\|_{(N_q), K, p_2}
\end{aligned}$$

where $p_2 \geq \max\{pH, p_1 H\}$, and where we use the inequality

$$\frac{1}{N_{|\alpha|} N_{|\beta|}} \leq \frac{AH^{|\alpha+\beta|}}{N_{|\alpha+\beta|}}$$

which follows from (M.2).

We shall now prove that for a fixed ϕ the mapping $g \longmapsto \psi$ defined by (12) is continuous. We shall prove that for every $p \in \boldsymbol{N}$ and $\varepsilon > 0$ there is a neighbourhood V of zero in $\mathscr{D}_y^{\prime(N_q)}$, such that if $g \in V$, then

$$\|\psi\|_{(N_q), K_1, p} < \varepsilon .$$

The set

$$B = \left\{ \theta(y);\ \theta(y) = \frac{p^{|\alpha|}}{N_{|\alpha|}} D_x^\alpha \phi(x, y);\ x \in K_1,\ \alpha \in \boldsymbol{N}_0^s \right\}$$

is bounded in $D_y^{(N_q)}$. Namely, for every $x \in K_1$, $\alpha \in \boldsymbol{N}_0^s$ and $p_1 \in \boldsymbol{N}$ we have

$$\begin{aligned}
\sup_{\substack{y \in K_2 \\ \beta \in \boldsymbol{N}_0{}^t}} \left\{ \frac{p^{|\alpha|} p_1^{|\beta|}}{N_{|\alpha|} N_{|\beta|}} |D_x^\alpha D_y^\beta \phi(x, y)| \right\} &\leq \sup_{\substack{(x, y) \in K \\ (\alpha, \beta) \in \boldsymbol{N}_0{}^{s+t}}} \left\{ \frac{AH^{|\alpha+\beta|} p^{|\alpha|} p_1^{|\beta|}}{N_{|\alpha+\beta|}} |D^{(\alpha, \beta)} \phi(x, y)| \right\} \\
&\leq A \|\phi\|_{(N_q), K, p_2}
\end{aligned}$$

where $p_2 = \max\{pH, p_1 H\}$. We put

$$V=\{g; |\langle g, \theta\rangle| <\varepsilon, \theta\in B\}.$$

Evidently, V has the properties mentioned above.

The proof that the mapping (12) is separately continuous in the case of spaces $\mathscr{E}_{x,y}^{(N_q)}$ and $\mathscr{E}_y'^{(N_q)}$ is similar to the given one for the spaces $\mathscr{D}_{x,y}^{(N_q)}$ and $\mathscr{D}_y'^{(N_q)}$ so we omit it.

Let us prove the same assertion for the spaces $\sigma_{x,y}$ and σ_y'.

THEOREM 7. *Let $M_p(x, q)=C_{p,q}\exp(m_p(x))$, $\tilde{M}_p(y, q)=C_{p,q}\exp(\tilde{m}_p(y))$ and $\bar{M}_p(x, y, q)=C_{p,q}\exp(m_p(x)+\tilde{m}_p(y))$. Further, let $\phi(x, y)\in\sigma_{x,y}\{\bar{M}_p\}$ and $g\in\sigma_y'\{\tilde{M}_p\}$. The function ψ defined by*

$$x\longmapsto\psi(x)=\langle g(y), \phi(x, y)\rangle$$

belongs to $\sigma_x\{M_p\}$, and the mapping from $\sigma_{x,y}\times\sigma_y'$ into σ_x defined by

$$(g, \phi)\longmapsto\psi \tag{13}$$

is hypocontinuous.

Proof. The proof is similar to the proof of Theorem 6, that is to the proof of Theorem 6.10 from [2]. The main difference lies in the fact that $\operatorname{supp}\psi$ is not bounded, in general. We shall prove that ψ is continuous. We ought to prove that if $x_n\to x_0$ then $\phi(x_n, y)$ converges to $\phi(x_0, y)$ in $\sigma_y\{\tilde{M}_p\}$.

We denote by B_y the ball $B(0, \rho)$ of sufficiently large radius ρ and by Y the set $\boldsymbol{R}^t\backslash B_y$. If $y\in Y$ and $x\in\boldsymbol{R}^s$, then $(x, y)\in Z$ where we put $Z=R^{s+t}\backslash B_{x,y}$ and $B_{x,y}=B(0, \rho)$ in $\boldsymbol{R}^{s+t}$.

We have

$$\sup_{\substack{y\in B_y\\ \alpha\in N_0{}^t}}\{C_{p,|\alpha|}|D_y^\alpha(\phi(x_n, y)-\phi(x_0, y))|\}\le\sup_{\substack{y\in B_y\\ |\alpha|<m}}\{C_{p,|\alpha|}|D_y^\alpha(\phi(x_n, y)-\phi(x_0, y))|\}$$
$$+\sup_{\substack{y\in B_y\\ |\alpha|\ge m}}\{C_{p,|\alpha|}|D_y^\alpha(\phi(x_n, y)-\phi(x_0, y))|\}.$$

For every $\alpha\in\boldsymbol{N}_0^t$

$$|D^\alpha(\phi(x_n, y)-\phi(x_0, y))|\to 0\qquad\text{as}\quad x_n\to x_0$$

uniformly on any compact subset of $\boldsymbol{R}^t$. The element ϕ from $\sigma_{x,y}\{M_p\}$ have the following property:

$$\sup_{\substack{y\in B_y\\ |q|>m}}\{C_{p,|q|}|\phi^{(q)}(x, y)|\}\to 0\qquad\text{as}\quad m\to\infty.$$

Thus we obtain

$$\sup_{\substack{y\in B_y\\ \alpha\in N_0{}^t}}\{C_{p,|\alpha|}|D_y^\alpha(\phi(x_n, y)-\phi(x_0, y))|\}\to 0\qquad\text{as}\quad x_n\to x_0$$

and accordingly

$$\text{(14)} \qquad \sup_{\substack{y\in B_y\\ \alpha\in N_0^t}} \{\tilde{M}_p(y, |\alpha|)|D_y^\alpha(\phi(x_n, y)-\phi(x_0, y))|\}\to 0 \qquad \text{as} \quad x_n\to x_0 .$$

If $X=\{x_0, x_1, x_2\cdots\}$ and $x\in X$ we have

$$\sup_{\substack{y\in Y\\ \alpha\in N_0^t}} \{\tilde{M}_p(y, |\alpha|)|D^\alpha\phi(x, y)|\}\le \sup_{\substack{(x,y)\in Z\\ q\in N_0^{s+t}}} \{\bar{M}_p(x, y, |q|)|\phi^{(q)}(x, y)|\} .$$

The last supremum tends to zero if $\rho\to\infty$. So, using this fact, (14) and

$$\sup_{\substack{y\in R^t\\ \alpha\in N_0^t}} \{\tilde{M}_p(y, |\alpha|)|D^\alpha(\phi(x_n, y)-\phi(x_0, y)|\}\le \sup_{\substack{y\in B_y\\ \alpha\in N_0^t}} \{\tilde{M}_p(y, |\alpha|)|D^\alpha(\phi(x_n, y)$$

$$-\phi(x_0, y))|\}+\sup_{\substack{y\in Y\\ \alpha\in N_0^t}} \{\tilde{M}_p(y, |\alpha|)(|D^\alpha\phi(x_n, y)|+|D^\alpha\phi(x_0, y)|)\} ,$$

we obtain that if $x_n\to x_0$, then $\phi(x_n, y)\to\phi(x_n, y)$ in $\sigma_y\{\tilde{M}_p\}$.

By using the idea of the preceding part of the proof one can prove that for every $p\in N$

$$\sup_{\substack{y\in R^t\\ \alpha\in N_0^t}} \left\{\tilde{M}_p(y, |\alpha|)\left|\frac{1}{h}(D_y^\alpha\phi(x+he_j, y)-D_y^\alpha\phi(x, y))-\frac{\partial}{\partial x_j}D_y^\alpha\phi(x, y)\right|\right\}$$

$$=\sup_{\substack{y\in R^t\\ \alpha\in N_0^t}} \left\{\tilde{M}_p(y, |\alpha|)\left|\frac{\partial}{\partial x_j}D_y^\alpha\phi(x+\theta he_j, y)-\frac{\partial}{\partial x_j}D_y^\alpha\phi(x, y)\right|\right\}\to 0 \qquad \text{as} \quad h\to 0 ,$$

where $e_j=(0, \cdots, 1, \cdots, 0)$ and $0<\theta<1$. This implies that for any $\alpha\in N_0^s$, $\psi^{(\alpha)}$ exists and

$$\psi^{(\alpha)}(x)=\langle g(y), D_x^\alpha\phi(x,y)\rangle .$$

To prove that $\psi\in\sigma_x\{M_p\}$, we ought to use (C.5) and the estimate

$$|\langle g(y), \theta(y)\rangle|\le C\gamma_{p_1}(\theta) , \qquad \theta\in\sigma_y\{\tilde{M}_p\} ,$$

which holds for some C and p_1.

In this way we obtain that for a given $p\in N$ there are $p_1\in N$ and $L_{p,p_1}>0$ such that

$$\text{(15)} \qquad \gamma_p(\psi(x))\le L_{p,p_1}\gamma_{p_1}(\phi(x, y)) .$$

From (15) we obtain that the mapping (13) is separately continuous and therefore hypocontinuous.

For investigations of the direct product and the convolution the following two theorems are of interest. First, we remark that the linear hull of the set of polynomials on R^s is a dense subspace of $\mathscr{E}^{(N_q)}(R^s)$. This follows directly from [2] Theorem 7.3. Let us denote this linear hull by L.

THEOREM 8. *The space $D_x^{(N_q)} \otimes \mathscr{D}_y^{(N_q)}$ is a dense subspace of $D_{x,y}^{(N_q)}$ and $\mathscr{E}_{x,y}^{(N_q)}$.*

Proof. Let $\phi(x,y) \in \mathscr{D}_{x,y}^{(N_q)}$ and let $\{\phi_n(x,y)\}$ be a sequence from L which in $E_{x,y}^{(N_q)}$ converges to ϕ. If $\operatorname{supp}\phi \subset K_1 \times K_2$, where K_1 and K_2 are compact subsets of $\boldsymbol{R}^s$ and $\boldsymbol{R}^t$, we denote by $\eta_1(x)$ and $\eta_2(y)$ functions from $\mathscr{D}_x^{(N_q)}$ and $\mathscr{D}_y^{(N_q)}$ such that $\eta_1(x)=1$ on K_1 and $\eta_2(y)=1$ on K_2. Since

$$\phi_n(x,y)=\sum_{i=0}^{k_n} a_{n,i}\phi_{n,1,i}(x)\phi_{n,2,i}(y)$$

we put

$$\begin{aligned}\psi_n(x,y)&=\sum_{i=0}^{k_n} a_{n,i}\phi_{n,1,i}(x)\eta_1(x)\phi_{n,2,i}(y)\eta_2(y)\\ &=\sum_{i=0}^{k_n} a_{n,i}\psi_{n,1,i}(x)\psi_{n,2,i}(y)\,.\end{aligned}$$

The sequence $\psi_n(x,y)$ is from $\mathscr{D}_x^{(N_q)} \otimes \mathscr{D}_y^{(N_q)}$ which converges to $\phi(x,y)$ in $\mathscr{D}_{x,y}^{(N_q)}$. We shall prove this.

If $K_3=\operatorname{supp}\eta_1$, $K_4=\operatorname{supp}\eta_2$ and $p\in\boldsymbol{N}$ we have

$$\text{(16)}\quad \sup_{\substack{(x,y)\in K_3\times K_4\\ q\in \boldsymbol{N}_0{}^{s+t}}}\left\{\frac{p^{|q|}}{N_{|q|}}|D^q(\phi(x,y)-\psi_n(x,y))|\right\} \le \sup_{\substack{(x,y)\in K_1\times K_2\\ q\in \boldsymbol{N}_0{}^{s+t}}}\left\{\frac{p^{|q|}}{N_{|q|}}|D^q(\phi(x,y)-\psi_n(x,y))|\right\}$$

$$+\sup_{\substack{(x,y)\in K_3\times K_4\backslash K_1\times K_2\\ q\in \boldsymbol{N}_0{}^{s+t}}}\left\{\sum_{\alpha\le q}\binom{q}{\alpha}\frac{A^{-1}H^{-|q|}p^{|q|}}{N_{|\alpha|}N_{|q-\alpha|}}|D^{q-\alpha}\phi_n(x,y)||D^\alpha(\eta_1(x)\eta_2(y))|\right\}.$$

The first supremum on the right hand side tends to zero when $n\to\infty$ because $\psi_n(x,y)=\phi_n(x,y)$ on $K_1\times K_2$. Let us prove the same for the second supremum on the right hand side.

$$\begin{aligned}\sup_{\substack{(x,y)\in K_3\times K_4\backslash K_1\times K_2\\ q\in \boldsymbol{N}_0{}^{s+t}}}&\left\{2^{-|q|}\sum_{\alpha\le q}\binom{q}{\alpha}\frac{(p/(2H))^{|q-\alpha|}}{N_{|q-\alpha|}}|D^{q-\alpha}\phi_n(x,y)|\right.\\ &\left.\times\frac{(p/(2H))^{|\alpha|}}{N_{|\alpha|}}|D^\alpha(\eta_1(x)\eta_2(y))|\right\}\le \|\phi_n(x,y)\|_{(N_q),\bar{K},p_1}\\ &\times\|\eta_1(x)\eta_2(y)\|_{(N_q),K,p_1}\end{aligned}$$

where $\bar{K}=\overline{(K_3\times K_4)\backslash(K_1\times K_2)}$ and $p_1\ge p/(2H)$. Since $\|\eta_1(x)\eta_2(y)\|_{(N_q),\bar{K},p_1}$ is bounded (for the proof of this fact we have to use (M.2) again) and $\|\phi_n(x,y)\|_{(N_q),\bar{K},p_1}\to 0$ as $n\to\infty$, (because $\phi(x,y)=0$ on $\bar{K}$) we obtain (16).

The space $\mathscr{D}_{x,y}^{(N_q)}$ is a dense subspace of $\mathscr{E}_{x,y}^{(N_q)}$. It implies that $\mathscr{D}_x^{(N_q)}\otimes\mathscr{D}_y^{(N_q)}$ is a dense subspace of $\mathscr{E}_{x,y}^{(N_q)}$.

We have proved that $\mathscr{D}_{x,y}^{(N_q)} \hookrightarrow \sigma_{x,y}\{M_p\}$, if condition (4) holds. So we directly obtain

THEOREM 9. *If* (4) *holds, then* $\mathscr{D}_x^{(N_q)} \otimes \mathscr{D}_y^{(N_q)} \hookrightarrow \sigma_{x,y}\{M_p\}$.

Now we shall define the direct product in the spaces $\mathscr{D}'^{(N_q)}$, $\mathscr{E}'^{(N_q)}$ and $\sigma'\{M_p\}$. Theorems 6 and 7 imply that the mapping

$$(17) \qquad (f(x), g(y)) \longmapsto f(x)g(y)$$

where $(f, g) \in \mathscr{D}_x'^{(N_q)} \times \mathscr{D}_y'^{(N_q)}$ (resp. $(f, g) \in \mathscr{E}_x'^{(N_q)} \times \mathscr{E}_y'^{(N_q)}$, resp. $(f, g) \in \sigma_x' \times \sigma_y'$) and $f(x)\cdot g(y)$ is defined by

$$(18) \qquad \langle f(x)g(y), \phi(x, y)\rangle := \langle f(x), \langle g(y), \phi(x, y)\rangle\rangle, \qquad \phi \in \mathscr{D}_{x,y}^{(N_q)}$$

$$(\text{resp. } \phi \in \mathscr{D}_{x,y}^{(N_q)}, \text{ resp. } \phi \in \sigma_{x,y})$$

is a separately continuous mapping from $\mathscr{D}_x'^{(N_q)} \times \mathscr{D}_y'^{(N_q)}$ (resp. $\mathscr{E}_x'^{(N_q)} \times \mathscr{E}_y'^{(N_q)}$, resp. $\sigma_x' \times \sigma_y'$) into $\mathscr{D}_{x,y}'^{(N_q)}$ (resp. $\mathscr{E}_{x,y}'^{(N_q)}$, resp. $\sigma_{x,y}'$).

THEOREM 10. *In the case of spaces* $\mathscr{D}'^{(N_q)}$, *the mapping* (17) *is hypocontinuous and in the cases of spaces* $E'^{(N_q)}$ *and* $\sigma'\{M_p\}$, *the mapping* (17) *is continuous.*

Proof. The second and third parts of the assertion follow from the fact that $\mathscr{E}'^{(N_q)}$ and $\sigma'\{M_p\}$ are strong duals of FS-spaces and Theorem 41.1 [7], p. 421.

Theorems 8 and 9 directly imply

THEOREM 11. *The operation of direct products in the spaces* $\mathscr{D}'^{(N_q)}$, $\mathscr{E}'^{(N_q)}$ *and* $\sigma'\{M_p\}$ *are commutative.*

In the usual way the direct product of n-elements from $\mathscr{D}'^{(N_q)}$, $\mathscr{E}'^{(N_q)}$ or $\sigma'\{M_p\}$ can be defined.

Let us remark that the direct product is associative.

Convolution in $\mathscr{D}'^{(N_q)}$

Up to the end of the paper we shall suppose that condition (4) holds.

We shall denote by $\{\eta_k(x, y); k \in \boldsymbol{N}\}$ the so-called unit sequence, if this sequence satisfies the following properties: $\eta_k(x, y) \in \mathscr{D}_{x,y}^{(N_q)}$, $k \in \boldsymbol{N}$; $|\eta_k^{(\alpha)}(x, y)| \le C_\alpha$, $\alpha \in \boldsymbol{N}_0^{s+t}$, $k \in \boldsymbol{N}$; for every compact set $K \subset \boldsymbol{R}^{s+t}$ there is k_0 such that $\eta_k(x, y) = 1$ on K for every $k > k_0$.

Let $f, g \in \mathscr{D}'^{(N_q)}(\boldsymbol{R}^s)$. If the limit

$$\lim_{k\to\infty} \langle f(x)g(y), \eta_k(x, y)\phi(x+y)\rangle, \qquad \phi \in \mathscr{D}_x^{(N_q)},$$

exists for any unit sequence (and therefore does not depend on the choice of a unit sequence) then from the completeness of the space $\mathscr{D}'^{(N_q)}$ it follows that the mapping

$$\phi \longmapsto \lim_{k\to\infty} \langle f(x)g(y), \eta_k(x, y)\phi(x, y)\rangle, \qquad \phi \in \mathscr{D}_x^{(N_q)},$$

defines an element from $\mathscr{D}'^{(N_q)}$ which is to be called the convolution of elements f and g and to be denoted by $f * g$.

This definition of convolution originates from Vladimirov's definition of the convolution of Schwartz distributions [8].

The convolution $f * g$, if exists at all, has the similar properties as the convolution of Schwartz distributions. The ideas for the proofs of the following theorem can be found in the book of Vladimirov [8] (see also [5]).

THEOREM 12.

(i) $(f * g)(x) = (g * f)(x)$;

(ii) $(f * g)^{(\alpha+\beta)}(x) = (f^{(\alpha)} * g^{(\beta)})(x)$;

(iii) $((f * g) * h)(x) = (f * (g * h))(x)$;

(iv) $(f * g)(x+h) = (f(t) * g(t+h))(x)$;

(v) *If* $f \in \mathscr{E}'^{(N_q)}$, *then* $f * g$ *exists for every* $g \in \mathscr{D}'^{(N_q)}$ *and* $\langle (f * g)(x), \phi(x) \rangle = \langle g(x), \langle f(t), \phi(t-x) \rangle \rangle$;

(vi) $(f(-t) * g(-t))(x) = (f(t) * g(t))(-x)$;

(vii) $f * \delta^{(j)} = f^{(j)}$;

(viii) $\operatorname{supp} f * g \subset \overline{\operatorname{supp} f + \operatorname{supp} g}$.

It follows from (v) that our definition of the convolution $f * g$ coincide with the definition of the convolution given in [2], p. 71, if $f \in \mathscr{E}'^{(N_q)}$ and $g \in \mathscr{D}'^{(N_q)}$.

If $f \in L^1_{\mathrm{loc}}$, the regular element from $\mathscr{D}'^{(N_q)}$ is defined by

$$\langle \tilde{f}, \phi \rangle = \int_{\boldsymbol{R}^s} f(x)\phi(x)dx\,, \qquad \phi \in \mathscr{D}^{(N_q)}.$$

If f and g are from L^1_{loc} and their ordinary convolution $|f| * |g| \in L^1_{\mathrm{loc}}$, we have that $\tilde{f} * \tilde{g}$ exists and $\tilde{f} * \tilde{g} = \widetilde{f * g}$.

Convolution in $\sigma'\{M_p\}$

We define the σ-convolution of elements f and g from $\sigma'\{M_p\}$ as the mapping from σ to $\boldsymbol{C}$ (set of complex numbers) by

$$\lim_{k \to \infty} \langle f(x)g(y), \eta_k(x, y)\phi(x+y) \rangle\,, \qquad \phi \in \sigma\,, \tag{19}$$

if this limit exists for every unit sequence $\{\eta_k\}$ (in this case this limit does not depend on the choice of a unit sequence). From the completeness of the space $\sigma'\{M_p\}$ according to the strong topology, it follows that if the limit in (19) exists for every unit sequence, then by (8) an element, let's name it $f \underset{\sigma}{*} g$, is defined.

It is clear that if $f, g \in \sigma'\{M_p\}$ and $f \underset{\sigma}{*} g$ exists then $f * g$ exists in $\mathscr{D}'^{(N_q)}$ and $f * g = f \underset{\sigma}{*} g$ on $\mathscr{D}^{(N_q)}$. This does not hold reversely. We shall give an example to show it.

Let $m_p(x)$, $p \in \boldsymbol{N}$, be the functions which satisfy the conditions from the begining of the paper and such that

$$m_p(x) = p\,|x|^{3/2} \qquad \text{for} \quad |x| > 1\,, \qquad x \in \boldsymbol{R}\,,$$

and let

$$C_{p,q} = \frac{p^q}{(q\,!)^\alpha}\,, \qquad \alpha > 1\,.$$

The function $\exp(-x^2)$ belongs to the corresponding space $\sigma\{M_p(x, q)\}$. Namely, using formulas from [3] p. 193 and p. 208 we have

$$|D^q \exp(-x^2)| = \exp(-x^2)H_n(x) \leq K \exp(-x^2/2)(q!)^{1/2}(2^{1/2})^q .$$

Thus the function $\exp x^3$ does not belong to $\sigma'\{M_p(x, q)\}$. But it was proved in [1] that for any non-negative function G there is a function ϕ which tends to zero when $|x| \to \infty$ such that the ordinary convolution $\phi * \phi$ exists and $\phi * \phi \geq G$.

Clearly, $\tilde{\phi}$ belongs to σ' and $\tilde{\phi} * \tilde{\phi}$ does not belong to σ'. So we obtain that $\tilde{\phi} \underset{\sigma}{*} \tilde{\phi}$ does not exist.

This motivates us to define the σ-convolution. In the forthcoming papers we shall investigate in details, convolution in the spaces and subspaces of ultradistributions and convolution equations in such spaces.

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